Complex Geometry Exercises

Week 8

Exercise 1.

(i) *Compute the signature of a closed complex surface* S *with* $\kappa(S) = -\infty$.

(ii) *Prove that* $\widetilde{S}_n = \#_n \mathbb{CP}^2$ *cannot be Kähler for* $n \geq 2$.

Exercise 2. *Consider the complex torus* $\mathbb{T}_\Lambda = \mathbb{C}^2/\Lambda$ *for a lattice* Λ.

(i) *Find* Λ *such that the only line bundle in* S_Λ *that admits a holomorphic structure is the trivial one.*

(ii) *Compute* $\mathrm{Pic}(\mathbb{T}_\Lambda)$.

(iii) *Prove (or convince yourself) that this is in fact true for a generic* Λ.

(iv) *Conclude that most tori in dimensions* ≥ 2 *are not projective.*

Exercise 3. *Let* X *be a complex manifold such that* $h^{2,0}(X) = 0$. *Show that* X *is projective. Conclude that Calabi-Yau manifolds (in the strict sense) of dimension* ≥ 3 *are all projective.*

Exercise 4. *Let* X *be a Kähler manifold, with* $\dim(X) \geq 4$, *and consider* $\iota : Y \to X$ *a hypersurface with* $\mathcal{O}(Y)$ *positive. Assuming* $H^2(X, \mathbb{Z})$ *and* $H^2(Y, \mathbb{Z})$ *are torsion-free, prove that* $\iota^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *is an isomorphism.*

Exercise 5. *Let* Σ *be a Riemann surface. Prove that*

$$c_1(K_\Sigma) = 2g(\Sigma) - 2 \ .$$

*(**Recall:** For a conformal metric* g, *the Gaussian curvature is* $K = \Delta \log(g)$.*)*

Exercise 6. *Let* Σ_d *be a smooth hypersurface in* $\mathbb{CP}^2$. *Show that*

$$g(\Sigma_d) = \frac{(d-1)(d-2)}{2} \ .$$